

Minimal Realizations of Supersymmetry for Matrix Hamiltonians

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Abstract

The notions of weak and strong minimizability of a matrix intertwining operator are introduced. Criterion of strong minimizability of a matrix intertwining operator is revealed. Criterion and sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian are presented. A method of constructing of a matrix Hamiltonian with a given constant symmetry matrix in terms of a set of arbitrary scalar functions and eigen- and associated vectors of this matrix is offered. Examples of constructing of 2×2 matrix Hamiltonians with given symmetry matrices for the cases of different structure of Jordan form of these matrices are elucidated.

1 Introduction

The matrix models with supersymmetry appear in Quantum Mechanics in several areas: in particular, for spectral design of potentials describing multichannel scattering and the motion of spin particles in external fields. The different cases of such models are considered in [1–16] and their systematic study was undertaken in [17–33] (see also the recent reviews [34,35]). In [17] intertwining of matrix Hermitian Hamiltonians by $n \times n$ first-order and 2×2 second-order matrix differential operators was investigated and the corresponding supersymmetric algebras were constructed. The main result of [18] is the formulae that help to build a modified $n \times n$ matrix Hamiltonian for a given $n \times n$ matrix (non-Hermitian, in general) Hamiltonian. The intertwining operator for this purpose is given by a $n \times n$ matrix linear differential operator of arbitrary order with the identity matrix coefficient at derivative d/dx in the highest degree that intertwines these Hamiltonians. The systematic study of intertwining relations for $n \times n$ matrix non-Hermitian, in general, one-dimensional Hamiltonians has been performed in [25,33] with intertwining realized by $n \times n$ matrix linear differential operators with nondegenerate coefficients at d/dx in the highest degree. Some methods of constructing of $n \times n$ matrix intertwining operator of the first order in derivative and of general form were proposed and their interrelations were examined.

In the one-dimensional QM with scalar Hamiltonians the isospectral transformations generally lead to a Nonlinear (Polynomial [36,37]) SUSY algebra. In respect to the SUSY partners there might be an infinite number of intertwining operators which provide the same pair of potentials but a different SUSY algebras [38,39]. These intertwining operators differ in factors which are functions of the Hamiltonians themselves. Thus for intertwining operators and the SUSY algebra itself the problem arises to minimize the order of their differential representation — minimizability problem. For some classes of potentials minimized algebras may exist which contain the same partner Hamiltonians but different sets of intertwining operators. In this case the hidden symmetry operators appear [38] being built of products of intertwining operators from different algebras. In the case of scalar Hamiltonians the number of independent minimized algebras cannot exceed two. The similar program has not been realized for matrix SUSY partner Hamiltonians although the important steps in this direction were done in [25].

In the present paper the following new results are elaborated:

- The notions of weak and strong minimizability of a matrix intertwining operator are introduced.
- Criterion of strong minimizability of a matrix intertwining operator is revealed.
- Criterion and sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian are presented.
- A method of constructing of a matrix Hamiltonian with a given constant symmetry matrix in terms of a set of arbitrary scalar functions and eigen- and associated vectors of this matrix is offered.
- Examples of constructing of 2×2 matrix Hamiltonians with given symmetry matrices for the cases of different structure of Jordan form of these matrices are elucidated.

The basic notations for intertwining operator algebra are defined in Sec.2. In Sec.3 we explain the motivation for introduction of the notions of weak and strong minimizabilities and give their definitions. The previous results on minimizability in scalar case are briefly formulated: in particular, the criterion of complete minimizability for a scalar intertwining operator from [38, 39]. At the end of this section the criterion of complete weak minimizability is given for a matrix intertwining operator from [25, 40] and the criterion of partial strong minimizability from the right for a matrix intertwining operator is presented. In Sec.4 the sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian is found. This condition provide us with the opportunity to receive the useful in practice formula for constructing of a matrix Hamiltonian with a given constant symmetry matrix. The two examples of constructing of arbitrary 2×2 matrix Hamiltonian with a given symmetry matrix A are presented. In the first of these examples there are two different eigenvalues for the matrix A and in the second of these examples normal (Jordan) form of A is a Jordan block. In Conclusion we outline the perspectives for further studies of the criteria of minimizabilities and hidden symmetries induced by extended SUSY algebras.

2 Basic definitions and notation

Let us consider two $n \times n$ matrix Hamiltonians defined on the entire axis,

$$H_+ = -I_n \partial^2 + V_+(x), \quad H_- = -I_n \partial^2 + V_-(x), \quad \partial \equiv d/dx,$$

where I_n is the identity matrix and $V_+(x)$ and $V_-(x)$ are square matrices, all elements of which are sufficiently smooth and, in general, complex-valued functions. We suppose that these Hamiltonians are *intertwined* by a matrix linear differential operator Q_N^- , so that

$$Q_N^- H_+ = H_- Q_N^-, \quad Q_N^- = \sum_{j=0}^N X_j^-(x) \partial^j, \quad (1)$$

where $X_j^-(x)$, $j = 0, \dots, N$ are also square matrices of n -th order, all elements of which are sufficiently smooth and, in general, complex-valued functions.

It is not hard to check that intertwining (1) leads to the following consequences:

$$X_N^-(x) = \text{Const}, \quad V_-(x) = X_N^- V_+(x) (X_N^-)^{-1} + 2X_{N-1}^{-\prime}(x) (X_N^-)^{-1}, \quad (2)$$

where and in what follows we restrict ourselves by the case $\det X_N^- \neq 0$.

Let's elucidate the structure of intertwining operator kernel and define the transformation vector-functions. By virtue of the intertwining the kernel of Q_N^- is an invariant subspace for H_+ :

$$H_+ \ker Q_N^- \subset \ker Q_N^-.$$

Hence, for any basis $\Phi_1^-(x), \dots, \Phi_d^-(x)$ in $\ker Q_N^-$, $d = \dim \ker Q_N^- = nN$ there is a constant square matrix $T^+ \equiv \|T_{ij}^+\|$ of d -th order such that

$$H_+ \Phi_i^- = \sum_{j=1}^d T_{ij}^+ \Phi_j^-, \quad i = 1, \dots, d.$$

A basis in the kernel of an intertwining operator Q_N^- in which the matrix T^+ has a normal (Jordan) form is called a *canonical basis*. Elements of a canonical basis are called a *transformation vector-functions*.

If a Jordan form of the matrix T^+ has block(s) of order higher than one, then the corresponding canonical basis contains not only formal vector-eigenfunctions of H_+ but also formal associated vector-function(s) of H_+ which are defined as follows [41].

A finite or infinite set of vector-functions $\Phi_{m,i}(x)$, $i = 0, 1, 2, \dots$ is called a *chain of formal associated vector-functions* of H_+ for a spectral value λ_m if

$$H_+ \Phi_{m,0} = \lambda_m \Phi_{m,0}, \quad \Phi_{m,0}(x) \neq 0, \quad (H_+ - \lambda_m I_n) \Phi_{m,i} = \Phi_{m,i-1}, \quad i = 1, 2, 3, \dots$$

3 Weak and strong minimizability of a matrix intertwining operator

Let us introduce the notions of weak and strong minimizabilities and give their definitions.

It is evident that if to multiply Q_N^- by a polynomial of the Hamiltonian,

$$Q_N^- \left[\sum_{l=0}^L A_l H_+^l \right] = \left(\left[\sum_{l=0}^L A_l H_-^l \right] Q_N^- \right),$$

where A_l , $l = 0, \dots, L$ are either a complex numbers or constant symmetry matrices for H_+ (H_-), then such products are again an intertwining operators for the same Hamiltonians:

$$\begin{aligned} \left\{ Q_N^- \left[\sum_{l=0}^L A_l H_+^l \right] \right\} H_+ &= Q_N^- H_+ \left[\sum_{l=0}^L A_l H_+^l \right] = H_- \left\{ Q_N^- \left[\sum_{l=0}^L A_l H_+^l \right] \right\} \\ \left(\left\{ \left[\sum_{l=0}^L A_l H_-^l \right] Q_N^- \right\} H_+ \right. &= \left. \left[\sum_{l=0}^L A_l H_-^l \right] H_- Q_N^- = H_- \left\{ \left[\sum_{l=0}^L A_l H_-^l \right] Q_N^- \right\} \right). \end{aligned}$$

Thus, the question arises about possibility to simplify an intertwining operator by separation from it a superfluous factor which is a polynomial of the corresponding Hamiltonian.

Let us now present the formal definitions of weak and strong minimizability of a matrix intertwining operator.

Definition 1. An intertwining operator Q_N^- is called *weakly minimizable* if this operator can be represented in the form

$$\begin{aligned} Q_N^- &= P_M^- \left[\sum_{l=0}^L a_l H_+^l \right] = \left[\sum_{l=0}^L a_l H_-^l \right] P_M^-, \\ a_l &\in \mathbb{C}, \quad l = 0, \dots, L, \quad a_L \neq 0, \quad 1 \leq L \leq \frac{N}{2}, \end{aligned} \quad (3)$$

where P_M^- is an $n \times n$ matrix linear differential operator of M -th order, $M = N - 2L$ that intertwines the Hamiltonians H_+ and H_- , so that $P_M^- H_+ = H_- P_M^-$. Otherwise, the operator Q_N^- is called *weakly non-minimizable*.

Definition 2. An intertwining operator Q_N^- is called *strongly minimizable from the right (from the left)* if this operator can be represented in the form

$$Q_N^- = P_M^- \left[\sum_{l=0}^L A_l H_+^l \right] \quad \left(Q_N^- = \left[\sum_{l=0}^L A_l H_-^l \right] P_M^- \right), \quad (4)$$

where A_l , $l = 0, \dots, L$, $1 \leq L \leq N/2$ are a symmetry $n \times n$ matrices for H_+ (H_-), $\det A_L \neq 0$ and P_M^- is an $n \times n$ matrix linear differential operator of M -th order, $M = N - 2L$ that intertwines the Hamiltonians H_+ and H_- , so that $P_M^- H_+ = H_- P_M^-$. Otherwise, the operator Q_N^- is called *strongly non-minimizable from the right (left)*.

Remark 1. In the scalar case $n = 1$ weak and strong minimizabilities are equivalent obviously. Thus, we shall speak below in the scalar case $n = 1$ about minimizability without additional specification “weak” or “strong”.

For comparison with our new results on strong minimizability of a matrix intertwining operator we present the previously formulated criterion of complete minimizability for a scalar intertwining operator [38, 39] and the criterion of complete weak minimizability for a matrix intertwining operator [25, 40]. The word “complete” means here that we consider such a minimization of an intertwining operator Q_N^- that the corresponding residual intertwining operator P_M^- in (3) is (weakly) non-minimizable.

Criterion of complete minimizability of a scalar intertwining operator.

In the scalar case $n = 1$ an intertwining operator Q_N^- can be represented in the form

$$Q_N^- = P_M^- \prod_{l=1}^s (\lambda_l - H_+)^{k_l}, \quad \lambda_l \in \mathbb{C}, \quad k_l \in \mathbb{N}, \quad l = 1, \dots, s, \quad \lambda_l \neq \lambda_{l'} \Leftrightarrow l \neq l',$$

where P_M^- is a non-minimizable linear differential operator of the M -th order that intertwines the Hamiltonians H_+ and H_- , so that $P_M^- H_+ = H_- P_M^-$,

if and only if

- (1) all numbers λ_l , $l = 1, \dots, s$ belong to the spectrum of the matrix T^+ and there are no equal numbers between them;
- (2) there are 2 Jordan blocks in a normal (Jordan) form of the matrix T^+ for any eigenvalue from the set λ_l , $l = 1, \dots, s$;
- (3) there are no 2 Jordan blocks in a normal (Jordan) form of T^+ for any eigenvalue of this matrix that does not belong to the set λ_l , $l = 1, \dots, s$;
- (4) k_l is the minimal of the orders of Jordan blocks corresponding to the eigenvalue λ_l in a normal (Jordan) form of the matrix T^+ , $l = 1, \dots, s$.

Criterion of complete weak minimizability of a matrix intertwining operator.

A matrix intertwining operator Q_N^- can be represented in the form

$$Q_N^- = P_M^- \prod_{l=1}^s (\lambda_l I_n - H_+)^{k_l}, \quad \lambda_l \in \mathbb{C}, \quad k_l \in \mathbb{N}, \quad l = 1, \dots, s, \quad \lambda_l \neq \lambda_{l'} \Leftrightarrow l \neq l',$$

where P_M^- is a non-minimizable matrix linear differential operator of the M -th order that intertwines the Hamiltonians H_+ and H_- , so that $P_M^- H_+ = H_- P_M^-$,

if and only if

- (1) all numbers λ_l , $l = 1, \dots, s$ belong to the spectrum of the matrix T^+ and there are no equal numbers between them;
- (2) there are $2n$ Jordan blocks in a normal (Jordan) form of the matrix T^+ for any eigenvalue from the set λ_l , $l = 1, \dots, s$;
- (3) there are no $2n$ Jordan blocks in a normal (Jordan) form of T^+ for any eigenvalue of this matrix that does not belong to the set λ_l , $l = 1, \dots, s$;
- (4) k_l is the minimal of the orders of Jordan blocks corresponding to the eigenvalue λ_l in a normal (Jordan) form of the matrix T^+ , $l = 1, \dots, s$.

Now let us give the criterion of partial strong minimizability from the right for a matrix intertwining operator. The word “partial” means here that we consider such minimization of a matrix intertwining operator Q_N^- that the corresponding residual intertwining operator P_M^- in (4) is not necessarily strongly non-minimizable from the right. As well we present in this section the interesting corollary of the mentioned above criterion that contains the criterion for a constant $n \times n$

matrix to be a symmetry matrix for an $n \times n$ matrix Hamiltonian.

Criterion of partial strong minimizability from the right of a matrix intertwining operator.

A matrix intertwining operator Q_N^- can be represented in the form

$$Q_N^- = P_{N-2}^-(A - H_+),$$

where A is a constant $n \times n$ matrix and P_{N-2}^- is a matrix linear differential operator of the order $N - 2$ that intertwines H_+ and H_- , so that $P_M^- H_+ = H_- P_M^-$, and the matrix A is a symmetry matrix for the Hamiltonian H_+ , $[H_+, A] = 0$,

if and only if

the kernel $\ker Q_N^-$ contains double set of associated vector-functions

$$\Phi_{ial}(x), \quad \Psi_{ial}(x), \quad i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 0, \dots, \nu_{ia} - 1,$$

$$\sum_{i=1}^m \sum_{a=1}^{g_i} \nu_{ia} = n, \quad \nu_{i1} \geq \nu_{i2} \geq \dots \geq \nu_{i,g_i}, \forall i$$

such that

(1)

$$\begin{aligned} H_+ \Phi_{ia0} &= \lambda_i \Phi_{ia0}, & (H_+ - \lambda_i I_n) \Phi_{ial} &= \Phi_{ia,l-1}, \\ H_+ \Psi_{ia0} &= \lambda_i \Psi_{ia0}, & (H_+ - \lambda_i I_n) \Psi_{ial} &= \Psi_{ia,l-1}, \\ \lambda_i &\in \mathbb{C}, & i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 1, \dots, \nu_{ia} - 1, & \lambda_i = \lambda_{i'} \Leftrightarrow i = i'; \end{aligned}$$

(2)

$$\Phi_{ial}(x) = \sum_{s=0}^l \sum_{t=1}^{g_{i,s+\nu_{ia}-l-1}} \varphi_{ia,l-s,t}(x) X_{its}, \quad \Psi_{ial}(x) = \sum_{s=0}^l \sum_{t=1}^{g_{i,s+\nu_{ia}-l-1}} \psi_{ia,l-s,t}(x) X_{its},$$

where

(a) X_{ial} , $i = 1, \dots, m$, $a = 1, \dots, g_i$, $l = 0, \dots, \nu_{ia} - 1$ is a complete set of constant eigen- and associated vectors of the matrix A , so that

$$AX_{ia0} = \lambda_i X_{ia0}, \quad (A - \lambda_i I_n) X_{ial} = X_{ia,l-1},$$

$$i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 1, \dots, \nu_{ia} - 1;$$

(b) g_{il} is the total number of vectors X_{ial} with fixed $i = 1, \dots, m$ and $l = 0, \dots, \nu_{i1}$, so that

$$\sum_{l=0}^{\nu_{i1}} g_{il} = \sum_{a=1}^{g_i} \nu_{ia}$$

is the algebraic multiplicity of the eigenvalue λ_i of the matrix A , $i = 1, \dots, m$;

(c) $\varphi_{ialb}(x)$ and $\psi_{ialb}(x)$ for any i , a , l and b are a scalar functions;

(3) the Wronskian of all vector-functions $\Phi_{ial}(x)$ and $\Psi_{ial}(x)$ does not vanish on the entire axis.

In view of the latter criterion, relation (2) and the result of [25, 40] on constructing of a matrix intertwining operator from a set of associated vector-functions the following corollary is valid.

Corollary 1 (criterion of existence of a symmetry matrix for a matrix Hamiltonian).

A constant $n \times n$ matrix A is a symmetry matrix for an $n \times n$ matrix Hamiltonian H of Schrödinger form, $[H, A] = 0$,

if and only if

there is a double set of associated vector-functions

$$\Phi_{ial}(x), \quad \Psi_{ial}(x), \quad i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 0, \dots, \nu_{ia} - 1,$$

$$\sum_{i=1}^m \sum_{a=1}^{g_i} \nu_{ia} = n, \quad \nu_{i1} \geq \nu_{i2} \geq \dots \geq \nu_{i, g_i}, \forall i$$

of the Hamiltonian H such that this set satisfies conditions 1 – 3 of the Criterion of partial strong minimizability from the right of a matrix intertwining operator.

The following Section 4 contains more useful in practice sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian.

4 Existence of a constant symmetry matrix for a matrix Hamiltonian

Let us formulate the sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian. This condition provide us with the opportunity to receive the formula for constructing of a matrix Hamiltonian with a given constant symmetry matrix which is useful in practice.

Sufficient condition of existence of a constant symmetry matrix for a matrix Hamiltonian.

A constant $n \times n$ matrix A is a symmetry matrix for an $n \times n$ matrix Hamiltonian H of Schrödinger form, $[H, A] = 0$, if there is a set of associated vector-functions

$$\Phi_{ial}(x), \quad i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 0, \dots, \nu_{ia} - 1, \quad \sum_{i=1}^m \sum_{a=1}^{g_i} \nu_{ia} = n,$$

of the Hamiltonian H such that

- (1) this set satisfies conditions 1 and 2 of the Criterion of partial strong minimizability from the right of a matrix intertwining operator;
- (2) the Wronskian of all vector-functions $\Phi_{ial}(x)$ does not vanish on the entire axis.

This condition allows us to use the following method of constructing of a matrix Hamiltonian with a given constant symmetry matrix.

Arbitrary $n \times n$ matrix Hamiltonian H of Schrödinger form with a given constant symmetry $n \times n$ matrix A can be found with the help of the following formula,

$$H = -\partial^2 I_n + A + \Phi''(x) \Phi^{-1}(x),$$

where $\Phi(x)$ is an $n \times n$ matrix-valued function constructed from n vector-functions

$$\Phi_{ial}(x), \quad i = 1, \dots, m, \quad a = 1, \dots, g_i, \quad l = 0, \dots, \nu_{ia} - 1, \quad \sum_{i=1}^m \sum_{a=1}^{g_i} \nu_{ia} = n,$$

as from columns. In this case a scalar functions $\varphi_{ialb}(x)$ from decompositions

$$\Phi_{ial}(x) = \sum_{s=0}^l \sum_{t=1}^{g_i, s+\nu_{ia}-l-1} \varphi_{ia, l-s, t}(x) X_{its}$$

(see above) are arbitrary parameterizing functions such that the Wronskian of all vector-functions $\Phi_{ial}(x)$, i.e. $\det \Phi(x)$, does not vanish on the entire axis.

Let us elucidate the advantages of the above construction with two examples of constructing of arbitrary 2×2 matrix Hamiltonian with a given symmetry matrix A . In the first of these examples there are two different eigenvalues for the matrix A and in the second of these examples normal (Jordan) form of A is a Jordan block.

Example 1: Constructing of a matrix 2×2 Hamiltonian with a given symmetry matrix A that has two different eigenvalues.

If A is a given 2×2 matrix that has two different eigenvalues,

$$AX_i = \lambda_i X_i, \quad \lambda_i \in \mathbb{C}, \quad X_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 1, 2, \quad \lambda_2 \neq \lambda_1,$$

then arbitrary 2×2 matrix Hamiltonian H , for which A is a symmetry matrix, can be found as follows:

$$\Phi_i(x) = \varphi_i(x)X_i, \quad i = 1, 2, \quad \Phi(x) = \begin{pmatrix} \varphi_1(x)x_{11} & \varphi_2(x)x_{21} \\ \varphi_1(x)x_{12} & \varphi_2(x)x_{22} \end{pmatrix},$$

$$W[\Phi_1(x), \Phi_2(x)] \equiv \det \Phi(x) = \varphi_1(x)\varphi_2(x)\Delta, \quad \Delta = x_{11}x_{22} - x_{12}x_{21},$$

$$H = -\partial^2 I_2 + \frac{1}{\Delta} \left(\frac{\varphi_1''}{\varphi_1} + \lambda_1 \right) \begin{pmatrix} x_{11}x_{22} & -x_{11}x_{21} \\ x_{12}x_{22} & -x_{12}x_{21} \end{pmatrix} + \frac{1}{\Delta} \left(\frac{\varphi_2''}{\varphi_2} + \lambda_2 \right) \begin{pmatrix} -x_{12}x_{21} & x_{11}x_{21} \\ -x_{12}x_{22} & x_{11}x_{22} \end{pmatrix}, \quad (5)$$

$$A = \frac{\lambda_1}{\Delta} \begin{pmatrix} x_{11}x_{22} & -x_{11}x_{21} \\ x_{12}x_{22} & -x_{12}x_{21} \end{pmatrix} + \frac{\lambda_2}{\Delta} \begin{pmatrix} -x_{12}x_{21} & x_{11}x_{21} \\ -x_{12}x_{22} & x_{11}x_{22} \end{pmatrix}, \quad (6)$$

where $\varphi_1(x)$ and $\varphi_2(x)$ are arbitrary smooth scalar functions without zeroes. The fact that $[H, A] = 0$ follows from evident commutativity of all matrices from the right-hand parts of (5) and (6).

Example 2: Constructing of a matrix 2×2 Hamiltonian with a given symmetry matrix A that has eigen- and associated vectors.

If A is a given 2×2 matrix that has eigen- and associated vectors,

$$AX_0 = \lambda_0 X_0, \quad (A - \lambda_0 I_2)X_1 = X_0, \quad \lambda_0 \in \mathbb{C}, \quad X_i = \begin{pmatrix} x_{i1} \\ x_{i2} \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = 0, 1,$$

then arbitrary 2×2 matrix Hamiltonian H , for which A is a symmetry matrix, can be found as follows:

$$\Phi_0(x) = \varphi_0(x)X_0, \quad \Phi_1(x) = \varphi_0(x)X_1 + \varphi_1(x)X_0, \quad \Phi(x) = \begin{pmatrix} \varphi_0 x_{01} & \varphi_0 x_{11} + \varphi_1 x_{01} \\ \varphi_0 x_{02} & \varphi_0 x_{12} + \varphi_1 x_{02} \end{pmatrix},$$

$$W[\Phi_0(x), \Phi_1(x)] \equiv \det \Phi(x) = \varphi_0^2(x)\Delta, \quad \Delta = x_{01}x_{12} - x_{02}x_{11},$$

$$H = -\partial^2 I_2 + \left(\frac{\varphi_0''}{\varphi_0} + \lambda_0 \right) I_2 + \frac{1}{\Delta} \left(\frac{\varphi_0'' \varphi_1 - \varphi_0 \varphi_1''}{\varphi_0^2} - 1 \right) \begin{pmatrix} x_{01}x_{02} & -x_{01}^2 \\ x_{02}^2 & -x_{01}x_{02} \end{pmatrix}, \quad (7)$$

$$A = \lambda_0 I_2 - \frac{1}{\Delta} \begin{pmatrix} x_{01}x_{02} & -x_{01}^2 \\ x_{02}^2 & -x_{01}x_{02} \end{pmatrix}. \quad (8)$$

where $\varphi_0(x)$ and $\varphi_1(x)$ are arbitrary smooth scalar functions and φ_0 is without zeroes. The fact that A is a symmetry matrix for H is obvious in view of (7) and (8).

5 Conclusions

Let us outline some possible problems for future studies.

- (1) To find a criterion of complete strong minimizability for a matrix intertwining operator.
- (2) To elaborate methods of spectral design for matrix Hamiltonians with the help of matrix intertwining operators of arbitrary order and, in particular, to reveal a criterion for transformation vector-functions that provides required changes for the spectrum of the corresponding final matrix Hamiltonian with respect to the spectrum of an initial matrix Hamiltonian. For this purpose one could try to generalize Index Theorem and Lemma 4 of [39, 42] for the matrix case.
- (3) To study (in)dependence of matrix differential intertwining operators in the way analogous to one of [38] and, in particular, to define the notions of dependence and independence for these operators, to find a criterion of dependence for them and to solve the questions on maximal number of independent matrix differential intertwining operators and on a basis of such operators.

- (4) By analogy with [38, 43] to investigate properties of a minimal matrix differential hidden symmetry operator.
- (5) To study (ir)reducibility of matrix differential intertwining operators and, in particular, to classify irreducible and absolutely irreducible [25] matrix differential intertwining operators in the way analogous to one of [37, 44–56].

Acknowledgments

This work was supported by RFBR grant 13-01-00136-a. The authors acknowledge Saint-Petersburg State University for a research grant 11.38.660.2013. AVS is grateful also to organizers of PHHQP 12 for hospitality and to Saint-Petersburg State University for a travel grant 11.46.1849.2013.

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